(1) moved deadline to 10/8

e := x, y, z | (\x -> e) | (e1 e2)

param result func arg

Syntactic Sugar

instead of	we write
`\x -> (\y -> (\z -> e))`	`\x -> \y -> \z -> e`
`\x -> \y -> \z -> e`	`\x y z -> e`
`(((e1 e2) e3) e4)`	`e1 e2 e3 e4`

((x y) z)

(x (y z))

```
\x y -> y      -- A function that that takes two arguments
               -- and returns the second one...

(\x y -> y) apple banana -- ... applied to two arguments
```

(((\x -> (\y -> y)) apple) banana)

Semantics : What Programs Mean

How do I "run" / "execute" a λ-term?

redex

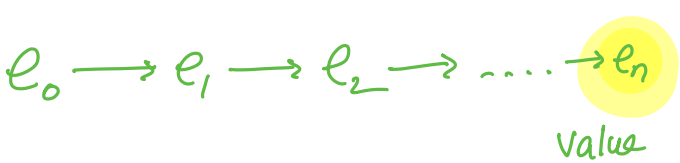
Think of middle-school algebra:

18

```
-- Simplify expression:
  (1 + 2) * ((3 * 8) - 2)
=
  3       * ((3 * 8) - 2)
=
  3       * ( 24     - 2)
=
  3       *  22
=
  66
```

Execute = rewrite step-by-step

- Following simple *rules*
- until no more rules *apply*



Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

But first we have to talk about **scope**

Semantics: Scope of a Variable

The part of a program where a **variable is visible**

In the expression `\x -> e`

- `x` is the newly introduced variable
- `e` is **the scope** of `x`
- any occurrence of `x` in `\x -> e` is **bound** (by the **binder** `\x`)

For example, `x` is bound in:

```
\x -> x
\x -> (\y -> x)
```

An occurrence of `x` in `e` is **free** if it's *not bound* by an enclosing abstraction

For example, `x` is free in:

```
x y                 -- no binders at all!
(\y -> x y)         -- no \x binder
(\x -> (\y -> y)) x -- x is outside the scope of the \x binder;
                    -- intuition: it's not "the same" x
```

QUIZ

In the expression `(\x -> x) x`, is `x` *bound* or *free*?

A. first occurrence is bound, second is bound

B. first occurrence is bound, second is free

C. first occurrence is free, second is bound

D. first occurrence is free, second is free

EXERCISE: Free Variables

An variable `x` is **free** in `e` if *there exists* a free occurrence of `x` in `e`

We can formally define the set of *all free variables* in a term like so:

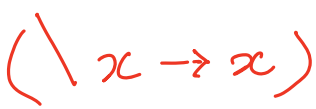
```
FV(x)       = ???
FV(\x -> e) = ???
FV(e1 e2)   = ???
```

Closed Expressions

If `e` has *no free variables* it is said to be **closed**

- Closed expressions are also called **combinators**

What is the shortest closed expression?

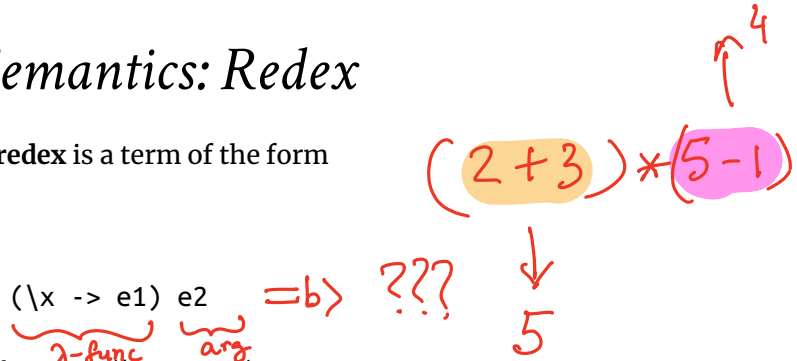


Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

Semantics: Redex

A **redex** is a term of the form



A *function* `(\x -> e1)`

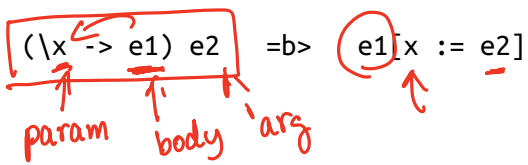
- `x` is the *parameter*
- `e1` is the *returned* expression

Applied to an argument `e2`

- `e2` is the *argument*

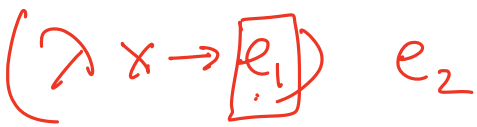
Semantics: β-Reduction

A **redex** b-steps to another term ...



where `e1[x := e2]` means

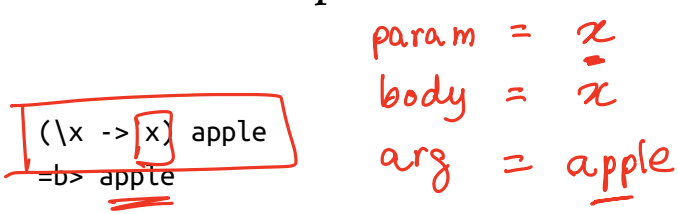
" `e1` with all *free* occurrences of `x` replaced with `e2` "



Computation by *search-and-replace*:

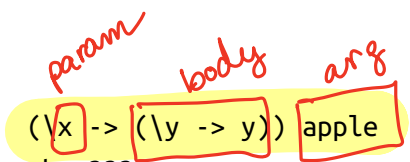
- If you see an *abstraction* applied to an *argument*, take the *body* of the abstraction and replace all free occurrences of the *formal* by that *argument*
- We say that `(\x -> e1) e2` β-steps to `e1[x := e2]`

Redex Examples



Is this right? Ask Elsa (http://goto.ucsd.edu:8095/index.html#?demo=blank.lc)!

QUIZ




```
=b> ???
```

A. `apple`

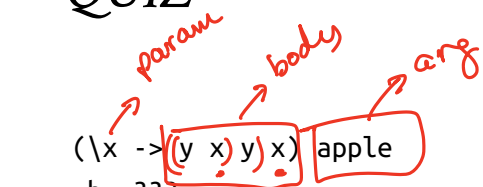
B. `\y -> apple`

C. `\x -> apple`

D. `\y -> y`

E. `\x -> y`

QUIZ



body[x := apple]

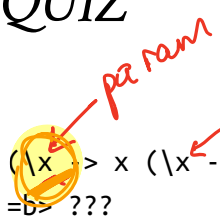
A. `(apple apple) apple apple`

B. `(y apple) y apple`

C. `(y y) y y`

D. `apple`

QUIZ



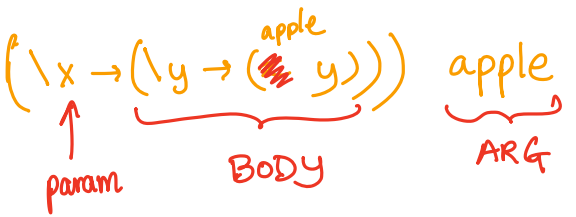
A. `apple (\x -> x)`

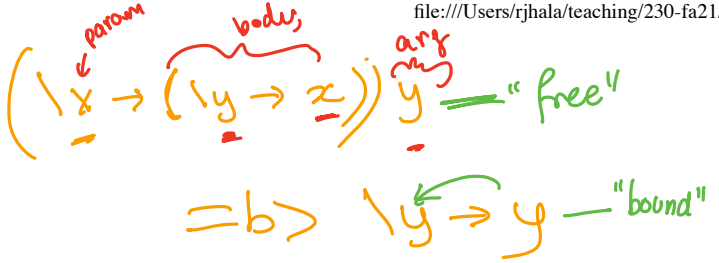
B. `apple (\apple -> apple)`

C. `apple (\x -> apple)`

D. `apple`

E. `\x -> x`





EXERCISE

What is a λ-term `fill_this_in` such that

```
fill_this_in apple
=b> banana
```

ELSA: https://goto.ucsd.edu/elsa/index.html

Click here to try this exercise (https://goto.ucsd.edu/elsa/index.html#?demo=permalink%2F1585434473_24432.lc)

A Tricky One


```
(\x -> (\y -> x)) y
=b> \y -> y
```

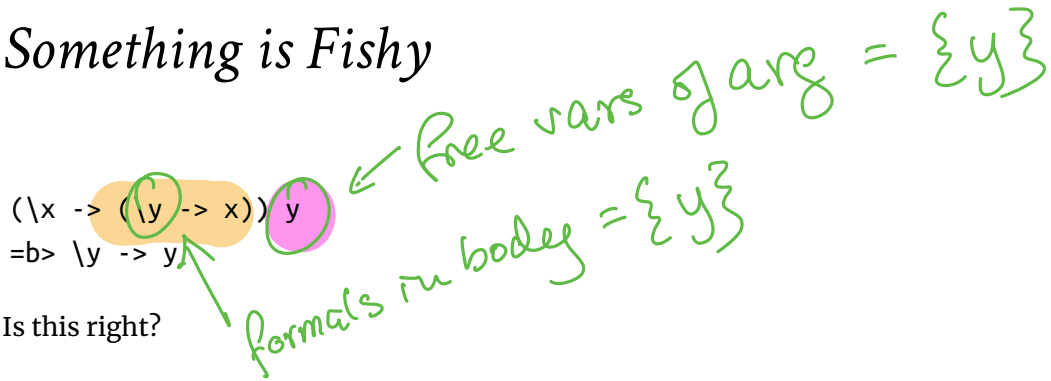
"free"

"bound"

:-C

Is this right?

Something is Fishy



Is this right?

Problem: The *free* `y` in the argument has been **captured** by `\y` in *body*!

Solution: Ensure that *formals* in the body are **different from** *free-variables* of argument!

function(y) { return y }

=a>

Capture-Avoiding Substitution

We have to fix our definition of β-reduction:

```
(\x -> e1) e2    =b>    e1[x := e2]
```

where `e1[x := e2]` means "~~`e1` with all *free* occurrences of `x` replaced with `e2`~~"

- `e1` with all *free* occurrences of `x` replaced with `e2`
- **as long as** no free variables of `e2` get captured

Formally:

```
x[x := e]           = e

y[x := e]           = y                          -- as x /= y

(e1 e2)[x := e]     = (e1[x := e]) (e2[x := e])

(\x -> e1)[x := e]  = \x -> e1                   -- Q: Why leave `e1
` unchanged?

(\y -> e1)[x := e]
  | not (y in FV(e)) = \y -> e1[x := e]
```

Oops, but what to do if `y` is in the *free-variables* of `e` ?

- i.e. if `\y -> ...` may *capture* those free variables?

Rewrite Rules of Lambda Calculus

1. β-step (aka *function call*)
2. α-step (aka *renaming formals*)

Semantics: α-Renaming

(\ x -> bob x)

free

(\ bob -> bob bob)

"bound"!

```
\x -> e  =a>  \y -> e[x := y]
  where not (y in FV(e))
```

y not free in e

- We rename a formal parameter `x` to `y`
- By replace all occurrences of `x` in the body with `y`
- We say that `\x -> e` α-steps to `\y -> e[x := y]`

Example:

```
\x -> x  =a>  \y -> y  =a>  \z -> z
```

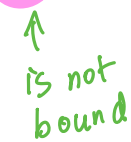
All these expressions are **α-equivalent**

What's wrong with these?

```
-- (A)
\f -> f x  =a>  \x -> x x
```

x is free in 'e'

```
-- (B)
(\x -> (\y -> y)) y  =a>   (\x -> (\z -> z)) z
```



Tricky Example Revisited

```
(\x -> (\y -> x)) y

                                  -- rename 'y' to 'z' to avoid capture
=a> (\x -> (\z -> x)) y

                                  -- now do b-step without capture!
=b> \z -> y
```

To avoid getting confused,

- you can **always rename** formals,
- so different **variables** have different **names**!

Normal Forms

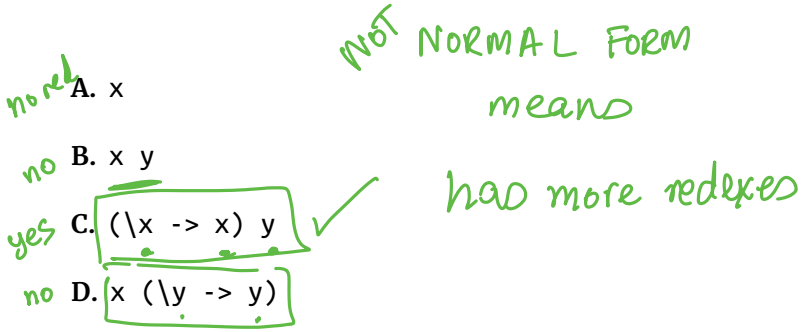
Recall **redex** is a λ-term of the form

```
(\x -> e1) e2
```

A λ-term is in **normal form** if it *contains no redexes*.

QUIZ

Which of the following term are **not** in *normal form* ?



Semantics: Evaluation

A λ-term `e` **evaluates to** `e'` if

1. There is a sequence of steps

```
e =?> e_1 =?> ... =?> e_N =?> e'
```

where each `=?>` is either `=a>` or `=b>` and `N >= 0`

2. `e'` is in *normal form*

Examples of Evaluation

```
(\x -> x) apple
  =b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)
  =?> ???
```

```
(\x -> x x) (\x -> x)
  =?> ???
```

Elsa shortcuts

Named λ-terms:

```
let ID = \x -> x  -- abbreviation for \x -> x
```

To substitute name with its definition, use a `=d>` step:

```
ID apple
  =d> (\x -> x) apple    -- expand definition
  =b> apple              -- beta-reduce
```

Evaluation:

- `e1 =*> e2` : `e1` reduces to `e2` in 0 or more steps
  - where each step is `=a>` , `=b>` , or `=d>`
- `e1 =~> e2` : `e1` evaluates to `e2` and `e2` is **in normal form**

EXERCISE

Fill in the definitions of `FIRST` , `SECOND` and `THIRD` such that you get the following behavior in `elsa`

```
let FIRST  = fill_this_in
let SECOND = fill_this_in
let THIRD  = fill_this_in

eval ex1 :
  FIRST apple banana orange
  =*> apple

eval ex2 :
  SECOND apple banana orange
  =*> banana

eval ex3 :
  THIRD apple banana orange
  =*> orange
```

ELSA: https://goto.ucsd.edu/elsa/index.html

Click here to try this exercise (https://goto.ucsd.edu/elsa/index.html#?demo=permalink%2F1585434130_24421.lc)

Non-Terminating Evaluation

```
(\x -> x x) (\x -> x x)
  =b> (\x -> x x) (\x -> x x)
```

e =b> e =b> . . . .

Some programs loop back to themselves...

... and *never* reduce to a normal form!

This combinator is called Ω

What if we pass Ω as an argument to another function?

```
let OMEGA = (\x -> x x) (\x -> x x)

(\x -> (\y -> y)) OMEGA
```

Does this reduce to a normal form? Try it at home!

Programming in λ-calculus

Real languages have lots of features

- Booleans
- Records (structs, tuples)
- Numbers
- **Functions** [we got those]
- Recursion

Lets see how to *encode* all of these features with the λ-calculus.