Example:

```
eval two_times_three :
  MULT TWO ONE
  =~> TWO
```

Programming in λ-calculus

- **Booleans** [done]
- **Records** (structs, tuples) [done]

- **Numbers** [done]
- **Lists**
- **Functions** [we got those]
- Recursion

λ-calculus: Lists

Lets define an API to build lists in the λ-calculus.

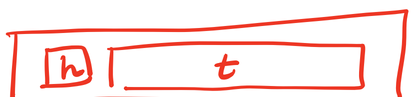
An Empty List

① Build

```
NIL
```

② Access

Constructing a list



A list with 4 elements

```
CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

intuitively `CONS h t` creates a *new* list with

- *head* `h`
- *tail* `t`

Destructing a list

- `HEAD l` returns the *first* element of the list
- `TAIL l` returns the *rest* of the list

```
HEAD (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
=~> apple

TAIL (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
=~> CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

λ-calculus: Lists

```
let NIL  = ???
let CONS = ???
let HEAD = ???
let TAIL = ???

eval exHd:
  HEAD (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
  =~> apple

eval exTl
  TAIL (CONS apple (CONS banana (CONS cantaloupe (CONS dragon NIL))))
  =~> CONS banana (CONS cantaloupe (CONS dragon NIL)))
```

CONS = PAIR
HEAD = FST
TAIL = SND

EXERCISE: Nth

Write an implementation of `GetNth` such that

- `GetNth n l` returns the `n-th` element of the list `l`

Assume that `l` has n or more elements

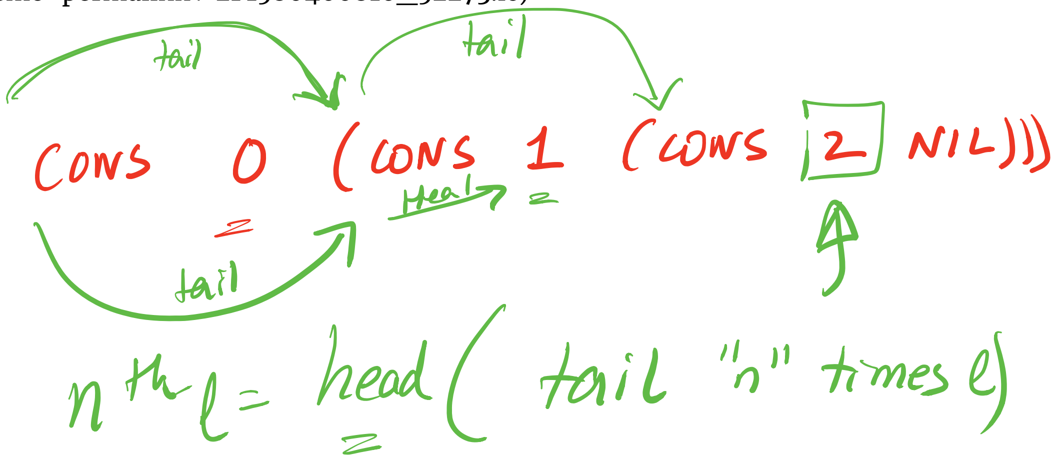
```
let GetNth = ???

eval nth1 :
  GetNth ZERO (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> apple

eval nth1 :
  GetNth ONE (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> banana

eval nth2 :
  GetNth TWO (CONS apple (CONS banana (CONS cantaloupe NIL)))
  =~> cantaloupe
```

Click here to try this in elsa (https://goto.ucsd.edu/elsa/index.html#?demo=permalink%2F1586466816_52273.lc)



λ-calculus: Recursion

I want to write a function that sums up natural numbers up to `n`:

```
let SUM = \n -> ...  -- 0 + 1 + 2 + ... + n
```

such that we get the following behavior

```
eval exSum0: SUM ZERO  =~> ZERO
eval exSum1: SUM ONE   =~> ONE
eval exSum2: SUM TWO   =~> THREE
eval exSum3: SUM THREE =~> SIX
```

0
0+1
0+1+2
0+1+2+3

Can we write sum **using Church Numerals**? ADD

Click here to try this in Elsa (https://goto.ucsd.edu/elsa/index.html#?demo=permalink%2F1586465192__52175.lc)

QUIZ

You *can* write `SUM` using numerals but its *tedious*.

Is this a correct implementation of `SUM`?

```
let SUM = \n -> ITE (ISZ n)
            ZERO
            (ADD n (SUM (DEC n)))
```

A. Yes

B. No

No!

- Named terms in Elsa are just syntactic sugar
- To translate an Elsa term to λ-calculus: replace each name with its definition

```
\n -> ITE (ISZ n)
        ZERO
        (ADD n (SUM (DEC n))) -- But SUM is not yet defined!
```

Recursion:

- Inside *this* function
- Want to call the *same* function on `DEC n`

Looks like we can't do recursion!

- Requires being able to refer to functions *by name*,
- But λ-calculus functions are *anonymous*.

Right?

λ-calculus: Recursion

Think again!

Recursion:

Instead of

- ~~Inside *this* function I want to call the *same* function on `DEC n`~~

Lets try

- Inside *this* function I want to call *some* function `rec` on `DEC n`
- And BTW, I want `rec` to be the *same* function

Step 1: Pass in the function to call "recursively"

```
let STEP =
  \rec -> \n -> ITE (ISZ n)
                  ZERO
                  (ADD n (rec (DEC n))) -- Call some rec
```

Step 2: Do some magic to `STEP`, so `rec` is itself

```
\n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))
```

That is, obtain a term `MAGIC` such that

```
MAGIC =*> STEP MAGIC
```

λ-calculus: Fixpoint Combinator

Wanted: a λ-term `FIX` such that

- `FIX STEP` calls `STEP` with `FIX STEP` as the first argument:

```
(FIX STEP) =*> STEP (FIX STEP)
```

(In math: a *fixpoint* of a function $f(x)$ is a point x, such that $f(x) = x$)

Once we have it, we can define:

```
let SUM = FIX STEP
```

Then by property of `FIX` we have:

```
SUM   =*>  FIX STEP  =*>  STEP (FIX STEP)  =*>  STEP SUM
```

and so now we compute:

```
eval sum_two:
  SUM TWO
  =*> STEP SUM TWO
  =*> ITE (ISZ TWO) ZERO (ADD TWO (SUM (DEC TWO)))
  =*> ADD TWO (SUM (DEC TWO))
  =*> ADD TWO (SUM ONE)
  =*> ADD TWO (STEP SUM ONE)
  =*> ADD TWO (ITE (ISZ ONE) ZERO (ADD ONE (SUM (DEC ONE))))
  =*> ADD TWO (ADD ONE (SUM (DEC ONE)))
  =*> ADD TWO (ADD ONE (SUM ZERO))
  =*> ADD TWO (ADD ONE (ITE (ISZ ZERO) ZERO (ADD ZERO (SUM DEC ZERO
)))
  =*> ADD TWO (ADD ONE (ZERO))
  =*> THREE
```

How should we define `FIX` ???

The Y combinator

Remember *Ω*?

```
(\x -> x x) (\x -> x x)
=b> (\x -> x x) (\x -> x x)
```

This is *self-replcating code*! We need something like this but a bit more involved…

The Y combinator discovered by Haskell Curry:

```
let FIX   = \stp -> (\x -> stp (x x)) (\x -> stp (x x))
```

How does it work?

```
eval fix_step:
  FIX STEP
  =d> (\stp -> (\x -> stp (x x)) (\x -> stp (x x))) STEP
  =b> (\x -> STEP (x x)) (\x -> STEP (x x))
  =b> STEP ((\x -> STEP (x x)) (\x -> STEP (x x)))
  --       ^^^^^^^^^^ this is FIX STEP ^^^^^^^^^^^
```

FIX STEP => STEP (FIX STEP)

SUM = FIX (\rec -> \n -> if (ISZ n) ZERO (ADD n (rec (DEC n))))

That's all folks, Haskell Curry was very clever.

Next week: We'll look at the language named after him (`Haskell`)

00-lam
01-trees
3 weeks

(https://ucsd-cse230.github.io/sp20/feed.xml) (https://twitter.com/ranjitjhala) (https://plus.google.com/u/0/104385825850161331469) (https://github.com/ranjitjhala)

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